# Nambu-Goldstone Fields, Anomalies and WZ Terms

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**Abstract**

We construct the Wess-Zumino terms from anomalies in case of quasigroups for the following situations. One is effective gauge field theories of Nambu-Goldstone fields associated with spontaneously broken global symmetries and the other is anomalous gauge theories. The formalism that we will develop can be seen as a generalization of the non-linear realization method of Lie groups. As an example we consider 2d gravity with a Weyl invariant regularization

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# 1 Introduction

Nambu-Goldstone bosons [1][2][3] appear in the low energy description of an underlying theory with a broken global symmetry $G$. The Goldstone bosons, $\theta^a$, are elements of the coset $\frac{G}{H}$, on which the symmetry group $G$ acts in a non-linear way. The dynamics of the Nambu-Goldstone bosons can be determined from the non-linear realization method[4][5][6]. The Lagrangian contains invariant terms and terms that are invariant up to total derivatives [7][8]. The number of quasi invariant terms is given by the cohomology group $H^{d+1}(\frac{G}{H}, R)$[9][10]

If one introduces fictitious gauge fields[1] into the underlying theory the global symmetry $G$ becomes local at the classical level but is broken quantum mechanically by anomalies $\mathcal{A}_a(\phi_g)$, where $\phi_g$ are the fictitious gauge fields. The gauge transformations associated with $\frac{G}{H}$ becomes anomalous while those associated with $H$ are non-anomalous. Since the underlying theory is anomaly free initially, these anomalies should be canceled by introducing spectator fields which are weakly coupled to the fictitious gauge fields so that the physical content of the theory is unchanged. We can have in mind, for example, a $SU(3) \times SU(3)$ broken theory with massless trapped fermions, in this case the spectator fields will be fermions that cancel the chiral gauge anomaly.

In the low energy description in order to have a vanishing anomaly [12], the gauged effective theory of Goldstone bosons must have an anomaly associated to the fictitious local symmetries. It is equal to the one produced in the underlying theory [7] i.e.,

$$\delta_a \mathcal{M}_1(\phi, \theta) = \mathcal{A}_a(\phi). \tag{1.1}$$

$\mathcal{M}_1(\phi, \theta)$ gives the interaction term between the Nambu-Goldstone fields and the fictitious gauge fields and is called the WZ term. It can also be used to study the interaction of Goldstone bosons coupled weakly to real gauge fields, for example the Goldstone bosons of the standard model interacting with the weak-electromagnetic gauge fields.

Another physical situation where (1.1) appears is in the case of anomalous gauge theories [13][14][15]. In these theories there are gauge degrees of freedom, $\theta$, that become propagating at quantum level. They are introduced in order to cure the initial anomalous gauge theory. The dynamics of these extra variables, for which there is no dynamics at classical level, is governed by the WZ term $\mathcal{M}_1(\phi, \theta)$. It differs from the case of broken global theories where the introduction of these fields causes no physical change of the initial theory.

In this paper we will consider a framework that could be applied to study both the WZ terms for the gauged effective theories of Goldstone bosons and anomalous gauge theories.[2] The construction of a dynamical action for NG fields from the anomalies, (1.1), has been investigated in the literature for Yang-Mills type gauge theories [7][8][16][17][18]. For more general gauge theories with soft and/or reducible open gauge algebras [19][20][21][22] we have constructed the WZ terms using an extended field-antifield formalism for the case in which the anomalous transformations form a closed algebra [23][24][25][26][27]. Here we

[1] Here we are following the nice discussion of Weinberg in ref [11]

[2] From here on we call the ordinary Nambu-Goldstone fields and the extra variables introduced to quantize anomalous gauge theories as Nambu-Goldstone (NG) fields.

will analyze a more general situation in which each anomalous and non-anomalous transformations do not form closed algebras. In this case we need, in general, to introduce NG fields for all the anomalous and non-anomalous transformations[3]. The presence of the NG fields associated to the non-anomalous transformations will generate gauge symmetries (hidden symmetries) of the WZ terms. Our analysis can be seen as a generalization of the non-linear realization method of Lie groups.

As an example we consider the case of a true anomalous gauge theory, the two dimensional gravity [13][29] with a Weyl invariant regularization [30][31]. In this case the Weyl and the area-preserving diffeomorphism transformations (SDiff), are non-anomalous. The anomalous transformations are other independent combination of the two dimensional diffeomorphism ($\mathrm{Diff}_2$). The local splitting into anomalous and non-anomalous transformations is not possible in this case. The WZ term constructed in terms of the original NG fields associated to $\mathrm{Diff}_2$ is non-local. However, in a natural way we can find a non-local changes of coordinates to a new set of NG fields such that the WZ term has a local form and coincides with the one introduced recently in the literature [30][31]. The non-local change of coordinates will give us a NG field parametrizing the coset, $\frac{Diff_2}{SDiff}$.

This paper is organized as follows. In section 2, we derive a general expression of the WZ terms and discuss the relation with the non-linear realization method. In section 3 the WZ action is constructed for the 2D gravity in a Weyl and SDiff invariant form. In section 4 we give a short summary and conclusions. There are three appendices devoted to more technical details.

# 2 WZ terms

In this section we will construct the antifield independent part of the WZ terms for general gauge theories with irreducible and closed algebras.[4] The gauged effective field theories of NG bosons and anomalous gauge theories have anomalous transformations that in general do not form a sub-group. For quasigroups the non-anomalous transformations are not closed also in general. In case of Lie groups the NG fields $\theta^a$ are elements of the coset $G/H$ [4][5][6] , with $H$ being the non-anomalous sub-group, and the WZ terms are constructed in ref.[7][8][16][17][9][10][18]. In more general cases we should introduce NG fields associated with the *non-anomalous* gauge transformations in addition to ones associated to the anomalous transformations, the presence of these NG fields will make the WZ terms invariant under new gauge transformations of NG fields.

## 2.1 Anomalous and Non-anomalous Gauge Transformations

Let us consider an action $S_0(\phi)$ with a irreducible gauge structure $\mathcal{G}$, that we assume to be closed off shell for simplicity. In the case of global spontaneously broken theories $S_0(\phi)$ describes the underlying gauge theory of the gauged effective field theory of NG bosons and in the case of anomalous gauge theories $S_0(\phi)$ is the classical action. In both

[3]For YM type theories this possibility has been studied for example in [16][28].

[4] The generalizations to reducible and on-shell theories may be examined by consulting [26].

cases the effective actions $\Gamma$ are not BRST invariant under all classical symmetries but we have anomalies $\mathcal{A}$;

$$\delta\Gamma = i\hbar\mathcal{A}. \tag{2.1}$$

At one loop it has a form[5]

$$\mathcal{A} \;=\; \mathcal{A}_\mu(\phi)c^\mu, \tag{2.2}$$

where $\mathcal{A}_\mu$'s are local functions of the classical fields $\phi^j$ and $c^\mu$'s are ghosts associated with the gauge transformations of $\mathcal{G}$. Some of them may be identically zero and some are linearly dependent. It may be expressed in terms of independent components $\tilde{\mathcal{A}}_a$ as

$$\mathcal{A}_\mu(\phi)c^\mu = \tilde{\mathcal{A}}_a(\phi)\tilde{Z}^a_\mu(\phi)c^\mu. \tag{2.3}$$

We can define therefore a subset of non-anomalous transformations as the ones whose infinitesimal parameters (ghosts) obey the system of local (differential) equations

$$\tilde{Z}^a_\mu(\phi)\epsilon^\mu = 0. \tag{2.4}$$

The solutions are expressed as

$$\epsilon^\mu \;=\; Z^\mu_A\;\tilde{\epsilon}^A, \qquad \text{with} \qquad \tilde{Z}^a_\mu\;Z^\mu_A \;=\; 0, \tag{2.5}$$

where $\tilde{\epsilon}^A$'s are the parameters of infinitesimal non-anomalous transformations . Here and hereafter we use the indices $a, b, ...$ as the anomalous components, $A, B, ...$ as the non-anomalous ones and $\mu, \nu, ...$ for all of them [6].

Since $\tilde{Z}^a_\mu$'s are local operators $Z^\mu_A$'s are chosen to be local. The infinitesimal anomalous transformations are ones that do not verify (2.4). In general the infinitesimal gauge transformations could be parametrized as

$$\epsilon^\mu \;=\; Z^\mu_a\;\tilde{\epsilon}^a + Z^\mu_A\;\tilde{\epsilon}^A, \tag{2.6}$$

here $\tilde{\epsilon}^a$'s are the anomalous and $\tilde{\epsilon}^A$'s are the non-anomalous parameters and $(Z^\mu_a, Z^\mu_A)$ and $(\tilde{Z}^\mu_a, \tilde{Z}^\mu_A)$ are dual basis of the orthogonal decomposition,

$$\tilde{Z}^b_\mu\;Z^\mu_a \;=\; \delta^b_a, \qquad \tilde{Z}^B_\mu\;Z^\mu_A \;=\; \delta^B_A, \qquad \text{thus} \qquad Z^\mu_a\;\tilde{Z}^a_\nu \;+\; Z^\mu_A\;\tilde{Z}^A_\nu = \;\delta^\mu_\nu. \tag{2.7}$$

$\tilde{Z}^a_\mu(\phi)$ and $Z^\mu_A(\phi)$'s are local operators but $\tilde{Z}^A_\mu(\phi)$ and $Z^\mu_a(\phi)$'s are in general non-local since they verify the inhomogeneous differential equations (2.7). Thus the splitting of the parameters (2.6) into the anomalous and non-anomalous ones is not performed locally in general. Since we should work in the space of local functionals we will not use the explicit decomposition (2.6); it also happens that the covariance prevents the explicit decomposition, as will be seen in the example of next section. Here we should point out that we can still consider *in a local way the non-anomalous transformations* defined in (2.4).

[5] For anomalies appearing in perturbative calculations, in case of closed theories, we can study the antifield independent part separately from the antifield dependent part, see ref [24]

[6] The De Witt condensed sum conventions for repeated indices are also understood.

## 2.2 Derivation of the WZ term

The WZ term for a gauged effective field theories of NG bosons or for anomalous gauge theories is a local functional satisfying $\delta\mathcal{M}_1 = i\mathcal{A}$ with the anomaly satisfying the WZ consistency condition $\delta\mathcal{A} = 0$. The anti-field independent part is found as a functional of the fields $\phi^i$ and NG fields $\theta^\mu$ and verifies

$$\delta_\mu \mathcal{M}_1(\phi, \theta^\nu) = i\mathcal{A}_\mu(\phi). \tag{2.8}$$

Remember that some of $\mathcal{A}_\mu$ may be identically zero and some are dependent. The solution of (2.8) is found in the extended space of $\phi^j$ and $\theta^\nu$ as a sum of a particular solution and the general solution of the homogeneous equation. A particular solution, in a form of non-local functional $\mathcal{M}_1^{non}(\phi)$, always exists since the BRST cohomology is trivial in the space of non-local functionals. In physical terms there always exists a non-local counter term which cancels the anomaly. It is an invariant function under non-anomalous transformations of $\phi^j$. The solution of the homogeneous equation is obtained by making use of the finite gauge transformation $F^j(\phi^i, \theta^\mu)$ of $\phi^j$ under $\mathcal{G}$. The NG fields $\theta^\mu$ are functions parametrizing the finite transformations. Some general properties of quasigroup structures are summarized in Appendix A.

The general solution of (2.8) is

$$\mathcal{M}_1(\phi, \theta^\nu) = \mathcal{M}_1^{non}(\phi) + G(F^j(\phi, \theta^\nu)) \tag{2.9}$$

if the transformation of $\theta^\nu$ is determined in such a way that

$$\delta_\epsilon F^j(\phi^i, \theta^\mu) = 0. \tag{2.10}$$

Since the fields transform as $\delta_\epsilon \phi^j = R^j_\nu(\phi)\epsilon^\nu$, the transformation of the NG fields is deduced, using (A.9) of Appendix A, as

$$\delta_\epsilon \theta^\nu = \; -\tilde{\mu}^\nu_\sigma(\theta; \phi)\epsilon^\sigma, \tag{2.11}$$

where $\tilde{\mu}^\nu_\sigma(\theta; \phi)$ is defined using the composition function $\varphi^\nu(\theta^\mu, \theta'^\mu; \phi^i)$ of the two transformations in $\mathcal{G}$ as in (A.10)

$$\tilde{\mu}^\nu{}_\sigma(\theta; \phi) \;=\; \left.\frac{\partial \varphi^\nu(\theta', \theta; \phi)}{\partial \theta'^\sigma}\right|_{\theta'=0}. \tag{2.12}$$

On the other hand the WZ term should verify the one-cocycle condition [7]

$$\mathcal{M}_1(\phi^j, \varphi^\nu(\theta, \theta'; \phi)) \;=\; \mathcal{M}_1(F^j(\phi, \theta), \theta'^\nu) \;+\; \mathcal{M}_1(\phi^j, \theta^\nu). \tag{2.13}$$

It determines the arbitrary function $G(\phi)$ in (2.9) and the solution satisfying the one-cocycle condition is

$$\mathcal{M}_1(\phi^j, \theta^\nu) = \mathcal{M}_1^{non}(\phi^j) - \mathcal{M}_1^{non}(F^j(\phi, \theta)). \tag{2.14}$$

Since we have introduced the NG fields for the non-anomalous transformations as well as for the anomalous ones the WZ term (2.14) is expected to have gauge invariances. In fact by taking the variation of the WZ term with respect to $\theta^\nu$, we have

$$\delta_\theta \mathcal{M}_1(\phi, \theta) = -\delta_\theta \mathcal{M}^{non}(F) = -i\mathcal{A}_\nu(F)\lambda^\nu{}_\sigma(\theta; \phi)\delta\theta^\sigma \tag{2.15}$$

where we have used the Lie equation (A.7) and $\lambda^{\nu}{}_{\sigma}(\theta;\phi)$ is an inverse of $\mu^{\sigma}{}_{\nu}(\theta;\phi)$ introduced in (A.8);

$$\mu^{\nu}{}_{\sigma}(\theta,\phi) = \left.\frac{\partial\varphi^{\nu}(\theta,\theta',\phi)}{\partial\theta'^{\sigma}}\right|_{\theta'=0} . \tag{2.16}$$

The WZ term $\mathcal{M}_1(\phi^j,\theta^{\mu})$ is invariant under

$$\delta_{\sigma}\theta^{\nu} \;=\; \xi^{\nu}_{A}(\theta;\phi)\sigma^{A} . \tag{2.17}$$

where $\sigma^A$'s are the parameters of the gauge transformation and $\xi^{\nu}_{A}$ is defined using the local operator $Z^{\rho}_{A}(\phi)$ in (2.5) by

$$\xi^{\nu}_{A}(\theta;\phi) \;\equiv\; \mu^{\nu}_{\rho}(\theta;\phi)Z^{\rho}_{A}(F(\theta;\phi)). \tag{2.18}$$

Actually the $F^i(\phi,\theta)$ transforms under (2.17) as a non-anomalous transformation,

$$\delta F^i(\phi,\theta) = R^i_{\mu}(F)Z^{\mu}_{A}(F)\sigma^A \tag{2.19}$$

and the WZ term (2.14) is invariant under arbitrary infinitesimal $\sigma^A$.

Combining (2.11) and (2.17), the transformation property of the NG fields $\theta^{\nu}$ is

$$\delta\;\theta^{\nu} = \;-\tilde{\mu}^{\nu}_{\mu}(\theta;\phi)\epsilon^{\mu} + \xi^{\nu}_{A}(\theta;\phi)\sigma^{A}. \tag{2.20}$$

Note the transformation properties of the NG fields is in general non-linear due to the structure of $\tilde{\mu}(\theta,\phi)$ and $\mu(\theta,\phi)$. Furthermore as we will see below the algebra of these transformations closes only on shell of the equations of motion of the WZ term.

Now let us rewrite the WZ term (2.14) as a surface integral in a variable $t$

$$\begin{aligned}\mathcal{M}_1(\phi^j,\theta^{\nu}) \qquad &= -\int_0^1 dt\,\frac{d}{dt}\mathcal{M}^{non}_1(F(\phi,\theta_t)) \\ &= -\int_0^1 dt\,\frac{\partial\mathcal{M}^{non}_1(F(\phi,\theta_t))}{\partial F^i}\frac{\partial F^i(\phi,\theta_t)}{\partial\theta^{\nu}_t}(\frac{\partial\theta^{\nu}_t}{\partial t})\end{aligned} \tag{2.21}$$

where $\theta^{\nu}(t)\equiv\theta^{\nu}_t$ is any interpolating function satisfying the boundary conditions

$$\theta^{\nu}(1) \;=\; \theta^{\nu}, \qquad \theta^{\nu}(0) \;=\; \theta^{\nu}_0 = \text{identity}\;\;\text{transformation}. \tag{2.22}$$

Using the Lie equation (A.7) we obtain

$$\mathcal{M}_1(\phi^j,\theta^{\nu}) = -\int_0^1 dt\;\left.\{\frac{\partial\mathcal{M}^{non}_1(\phi)}{\partial\phi^i}R^i_{\nu}(\phi)\}\right|_{\phi=F(\phi,\theta_t)}\lambda^{\nu}_{\rho}(\theta_t;\phi)(\partial_t\theta^{\rho}_t). \tag{2.23}$$

It can be expressed in terms of the anomaly as

$$\mathcal{M}_1(\phi^j,\theta^{\nu}) = \;-i\int_0^1 dt\,\mathcal{A}_{\nu}(F(\phi,\theta_t))\lambda^{\nu}{}_{\rho}(\theta_t;\phi)(\partial_t\theta^{\rho}_t). \tag{2.24}$$

Note that this expression is valid for a gauged effective theory of Goldstone bosons associated with a broken global symmetry and for anomalous gauge theories.

## 2.3 The Extended Action

In order to study the gauge structure of the extended formalism of the fields $\phi^i$ and $\theta^\mu$ in more detail it is useful to introduce a field-antifield formalism and the classical master equation (CME) that encodes all gauge structure of the theory [33][34]. The solution of CME is constructed in the extended phase space of $(\phi^j, c^\nu, \theta^\nu, b^A)$ and their anti-fields $(\phi^*_j, c^*_\nu, \theta^*_\nu, b^*_A)$. The original solution $S$ of CME, $(S,S)=0$, is

$$S \;=\; S_0(\phi^j) \;+\; \phi^*_j\; R^j_\nu(\phi)\; c^\nu \;+\; \frac{1}{2} c^*_\nu\; T^\nu_{\rho\sigma}(\phi)\; c^\sigma c^\rho, \tag{2.25}$$

where $T^\nu_{\rho\sigma}(\phi)$ is the structure function of $\mathcal{G}$ depending on $\phi$ in case of quasigroups. The extended action $\tilde{S}$ of the system is

$$\tilde{S} \;=\; S \;+\; \theta^*_\nu\; \{-\tilde{\mu}^\nu_\sigma(\theta;\phi)c^\sigma \;+\; \xi^\nu_A(\theta;\phi)b^A\} \;+\; \frac{1}{2} b^*_A \tilde{T}^A_{BD}(F(\phi,\theta))b^D b^B, \tag{2.26}$$

where

$$\tilde{T}^{\tilde{\rho}}_{\tilde{\mu}\tilde{\nu}}(\phi) \;=\; \tilde{Z}^{\tilde{\rho}}_\rho\; T^\rho_{\mu\nu}\; Z^\mu_{\tilde{\mu}}\; Z^\nu_{\tilde{\nu}} \;-\; \tilde{Z}^{\tilde{\rho}}_\rho\; \frac{\partial Z^\rho_{[\tilde{\nu}}}{\partial \phi^i}\; R^i_\sigma\; Z^\sigma_{\tilde{\mu}]}. \tag{2.27}$$

The second term of (2.26) comes from the transformation properties of the $\theta^\nu$ in (2.20). $b^A$ is ghost for the $\sigma^A$ gauge transformation of the WZ term. The last term comes from the algebra of the $\sigma^A$ transformations in (2.20). The derivations are given in the Appendix B. The $\tilde{S}$ satisfies

$$(\tilde{S},\tilde{S}) \;=\; -\{\tilde{\theta}^*_\mu\; \mu^\mu_\nu\; Z^\nu_a(F) \;+\; b^*_A\; \tilde{T}^A_{aB}(F)\; b^B\}\; \tilde{T}^a_{DE}(F) b^E b^D. \tag{2.28}$$

As will be shown the $\tilde{T}^a_{DE}(F)$ vanishes using the WZ equation of motion the $\tilde{S}$ verifies the CME only on shell. The spinning string with a super-diffeomorphism invariant regularization shows an example of this phenomena [27].

In order to see $\tilde{T}^a_{DE}(F)$ vanishes on shell[7] consider the fact that the effective action is invariant under non-anomalous transformations , i.e.,

$$\delta\Gamma \;=\; \frac{\partial \Gamma}{\partial \phi^i}\; R^i_\mu\; (\phi) Z^\mu_A(\phi)\tilde{\epsilon}^A \;=\; 0 \tag{2.29}$$

and therefore

$$0 \;=\; [\; \delta(\epsilon), \delta(\epsilon')\; ]\; \Gamma \;=\; \tilde{\mathcal{A}}_a(\phi)\tilde{T}^a_{AB}(\phi)\tilde{\epsilon}^A\tilde{\epsilon}'^B. \tag{2.30}$$

Thus $\tilde{T}^a_{AB}(\phi)$ is a linear combination of the anomalies with anti-symmetric coefficients $E^{ab}_{AB}(\phi) = -E^{ba}_{AB}(\phi)$,

$$\tilde{T}^a_{AB}(\phi) \;=\; E^{ab}_{AB}(\phi)\tilde{\mathcal{A}}_b(\phi). \tag{2.31}$$

On the other hand the equations of motion for the NG fields are seen from (2.15), using the fact that $\lambda$ is non-singular, as

$$\mathcal{A}_\mu(F) \;=\; 0. \tag{2.32}$$

[7]We acknowledge discussions with J.M. Pons and F. Zamora clarifying this point.

Thus $\tilde{T}^a_{AB}(F)$ vanishes on shell of WZ equation of motion.

So far we have introduced the NG fields $\theta^\mu$ for all transformations. It is not always necessary when there is a subgroup $\mathcal{G}_0$ of $\mathcal{G}$ and $\mathcal{G}_0$ contains all anomalous transformations . In this case the NG fields $\theta^J$ associated with the transformation in $\mathcal{G}-\mathcal{G}_0$ and the corresponding ghosts $b^J$ can be eliminated consistently. The reduction is proceeded by using a canonical transformation in Appendix C. The crucial point is that the BRST transformations for $\theta^J$ is nilpotent on the reduced space defined by $\theta^J=0$. As a result the transformation property of the remaining NG fields is modified from (2.20) to one in (C.12),

$$\delta\theta^\alpha \;=\; \{-\tilde{\mu}'^{\alpha}_{\nu}\; c^\nu \;+\; \xi^\alpha_A\; b^A\}, \qquad\qquad \tilde{\mu}'^{\alpha}_{I} \;\equiv\; \tilde{\mu}^\alpha_I - \mu^\alpha_J \lambda^J_L \tilde{\mu}^L_I, \quad \tilde{\mu}^{\alpha'}_{\beta} \;\equiv\; \tilde{\mu}^\alpha_\beta. \tag{2.33}$$

where indices $\alpha, \beta, ...$ are those of $\mathcal{G}_0$ and $I, J, ...$ are non-anomalous indices of $\mathcal{G}-\mathcal{G}_0$.

## 2.4 Relation to $G/H$ non-linear realization for Lie groups

If the theory we are considering is such that the non-anomalous transformations close between themselves, $\tilde{T}^a_{AB}=0$, (2.28) tells $\tilde{S}$ satisfies the CME off shell thus the commutators of the $\sigma$ transformations of the NG fields is closed. If furthermore we can split in a local way the anomalous and non-anomalous transformations we can eliminate the NG fields $\theta^A$ associated to the non-anomalous transformations explicitly. This is the case for YM Lie groups . As we will see our procedure gives a generalization of $G/H$ non-linear realization method of YM Lie groups to the case of quasigroups.

Let us restrict to the surface $\theta^A=0$. In order to have consistent transformations on this surface we should impose $\delta\theta^A|_{\theta^A=0}=0$. Using (2.20) the parameters of $\sigma$ transformations are determined in terms of the parameters $\epsilon^\mu$

$$\sigma^B = \Lambda^B_A(\theta^a, \phi^j)\; \tilde{\mu}^A_\nu(\theta^a, \phi^j)\; \epsilon^\nu \tag{2.34}$$

where $\tilde{\mu}^A_\nu(\theta^a, \phi^j) \equiv \tilde{\mu}^A_\nu(\theta^\nu, \phi^j)|_{\theta^A=0} = 0$ and $\Lambda^B_A$ is the inverse matrix of $\xi^A_B|_{\theta^A=0}$. From (2.20) and (2.34) we deduce the transformation law for the NG fields $\theta^a$ associated with the anomalous transformations as

$$\delta\theta^a = \;-\,[\tilde{\mu}^a_\nu(\theta^a, \phi^j) \;-\; \xi^a_A(\theta^a, \phi^j)\Lambda^A_B(\theta^a, \phi^j)\tilde{\mu}^B_\nu\;(\theta^a, \phi^j)]\;\epsilon^\nu. \tag{2.35}$$

This is a non-linear realization of the quasigroup in terms of the NG fields $\theta^a$ *and* $\phi^j$. In this case the WZ term is given as a function of $\phi$ and $\theta^a$, by putting $\theta^A=0$ as

$$\mathcal{M}_1(\phi, \theta^a) = -i\int_0^1 dt\, \mathcal{A}_\nu(F(\phi, \theta^a_t))\lambda^\nu{}_b(\theta^a_t;\phi)(\partial_t\theta^b_t). \tag{2.36}$$

There is no gauge symmetry in this WZ term.

In the reduced variables the one-cocycle condition corresponding to (2.13) holds also as

$$\mathcal{M}_1(\phi, \tilde{\varphi}^d(\theta^a, \theta'^b;\phi)) \;=\; \mathcal{M}_1(F(\phi, \theta^a), \theta'^b) \;+\; \mathcal{M}_1(\phi, \theta^a). \tag{2.37}$$

Here the composition of two anomalous transformations $\varphi(\theta^a, \theta'^b;\phi)$ produces non anomalous components $\varphi^A$ as well as anomalous one $\varphi^d$. However it is expressed as a composition

of an anomalous transformation $\tilde{\varphi}^d$ and successive non-anomalous one and since the WZ term is invariant under the non-anomalous transformations we obtain the result (2.37).

In the case of a YM Lie group $G$ with a reductive non-anomalous subgroup $H$, the formula (2.35) does not depend on the fields $\phi$ and reproduces the infinitesimal transformation of the NG fields $\theta^a$ of the coset $G/H$ obtained from the standard method of non-linear realizations [4][5]. In this case $\tilde{\mu}$ and $\mu$ respectively give the left and the right (infinitesimal) transformations of the group element. The group element $g$ is parametrized using some representation $t_\mu$ of $\mathcal{G}$ as $g = e^{\theta^\mu t_\mu}$. If we parametrize left transformations as $g_L = e^{-\epsilon^\mu t_\mu}$ and right transformations as $g_R = e^{\sigma^\mu t_\mu}$ then $g$ transforms to $g' = g_L\ g\ g_R$. If we define the composition function by $gg' = e^{\theta^\mu t_\mu} e^{\theta'^\mu t_\mu} = e^{\varphi(\theta,\theta')^\mu t_\mu}$ the infinitesimal transformation of $\theta^\nu$ is given by

$$\delta\theta^\rho \ = \ -\tilde{\mu}^\rho_\nu(\theta)\epsilon^\nu \ + \ \mu^\rho_\nu(\theta)\sigma^\nu, \tag{2.38}$$

where $\mu$ and $\tilde{\mu}$ are defined as (2.16) and (2.12) though they are independent of $\phi$.

The non-linear realization is defined by restricting $\theta^\rho$ to be the coset coordinates $\theta^a$, i.e. the NG fields, and are transformed under the left action of $G$ with the associative right action of $H$. The latter is determined so that the $\theta'^\rho$ remains in the coset. For infinitesimal transformations the condition is

$$0 \ = \ \delta\theta^A \ = \ \{-\tilde{\mu}^A_\nu(\theta)\epsilon^\nu \ + \ \mu^A_B(\theta)\sigma^B\}\Big|_{\theta^A=0}, \qquad \rightarrow \qquad \sigma^B = \Lambda^B_A\ \tilde{\mu}^A_\nu\ \epsilon^\nu \tag{2.39}$$

where $\Lambda^B_A$ is the inverse matrix of $\mu^A_B|_{\theta^A=0}$. It follows the transformation law for the NG fields $\theta^a$

$$\delta_\epsilon\theta^a = \ -[\tilde{\mu}_\nu{}^a(\theta^a) \ - \ \mu^a_A(\theta^a)\Lambda^A_B(\theta^a)\tilde{\mu}^B_\nu\ (\theta^a)]\ \epsilon^\nu \tag{2.40}$$

which is one corresponding to our general quasigroup result (2.35). Note in this case we always split the anomalous transformations and non-anomalous transformations thus the $Z$ and $\tilde{Z}$ in (2.7) are taken to be unit matrices and $\xi^\rho_A = \mu^\rho_A$. In the case the anomalous transformations from a subgroup the matrix $\Lambda^A_B(\theta^a)$ coincides with $\lambda^A_B(\theta^\nu)\Big|_{\theta^A=0}$.

Using the explicit forms of $\mu$ and $\tilde{\mu}$ in the ajoint representation

$$\mu^\rho_\sigma = (\frac{\hat{\theta}}{e^{\hat{\theta}}-1})^\rho_\sigma, \quad \tilde{\mu}^\rho_\sigma = (\frac{-\hat{\theta}}{e^{-\hat{\theta}}-1})^\rho_\sigma, \quad \hat{\theta}^\rho_\sigma \equiv \theta^\nu T^\rho_{\nu\sigma} \tag{2.41}$$

we can see how the transformations associated with $H$ produces linear transformations for the NG fields while the ones associated with the coset are non-linear.

# 3 Weyl Invariant 2D Gravity

In this section we apply the general formula developed in the previous section to the case of gravity in two dimensions. We use zwei-bein formalism for the gravitational fields. The classical gauge symmetries are thus the local Lorentz, the Weyl and the two dimensional diffeomorphism ($\text{Diff}_2$) transformations. The two dimensional gravity coupling to scalar matter (bosonic string) has been discussed as a system with anomalous Weyl symmetry [13]. This result comes from the regularization preserving the diffeomorphism

invariance. Recently a Weyl invariant formulation has been discussed[30][31][32]. The anomaly is

$$\mathcal{A} \;=\; k\int d^2x\; R(\frac{g_{\mu\nu}}{\sqrt{-g}})\partial_\alpha c^\alpha \;=\; \mathcal{A}_\alpha\; c^\alpha, \tag{3.1}$$

where $R(\frac{g_{\mu\nu}}{\sqrt{-g}})$ is a scalar curvature of the Weyl invariant metric. The diffeomorphism ghosts $c^\alpha, (\alpha = 0, 1)$ appear in a form of divergence and the components of anomaly $\mathcal{A}_\alpha$ are not independent, we see that the area preserving diffeomorphism (SDiff) is the non-anomalous transformation . The corresponding infinitesimal transformation parameters are satisfying $\partial_\alpha\epsilon^\alpha = 0$, they form a sub-algebra of the diffeomorphism transformations. Ones that do not satisfy $\partial_\alpha\epsilon^\alpha = 0$ generate anomalous transformations .

The splitting of the diffeomorphism transformations into the anomalous and non-anomalous transformations cannot be performed locally and covariantly in this case. In constructing the WZ term we take diffeomorphism as the $\mathcal{G}_0$, a subgroup including all anomalous transformations, we do not need to introduce the NG fields for the local Lorentz and the Weyl transformations. For $\mathrm{Diff}_2$ we consider finite coordinate transformations

$$x^\alpha \;\rightarrow\; \tilde{x}^\alpha \;=\; f^\alpha(x) \tag{3.2}$$

and the functions $f^\alpha(x),\;(\alpha = 0, 1)$ play the role of the NG fields $\theta^\alpha$. The corresponding finite transformation of zwei-bein is

$$e_\alpha^{\;a} \;\rightarrow\; F_\alpha^{\;a}(e, f) \;=\; \frac{\partial f^\beta(x)}{\partial x^\alpha}e_\beta^{\;a}(f(x)) \equiv\; A_\alpha^{\;\beta}(x)\; e_\beta^{\;a}(f(x)). \tag{3.3}$$

The determinant $e = \det e_\alpha^a$ is transformed as

$$e(x) \;\rightarrow\; \tilde{e}(x) \;=\; \Delta^f(x)e(f(x)), \qquad \Delta^f(x) \equiv \det A = \det\;(\partial_\alpha f^\beta(x)). \tag{3.4}$$

The successive two finite transformations gives the composition function $\varphi(f, f')$ as

$$\varphi^\alpha(f, f') \;=\; f''^\alpha(x) \;=\; f^\alpha(f'(x)). \tag{3.5}$$

$\tilde{\mu}^\alpha_\beta(f)$ in (2.12) has a non-local expression,

$$\tilde{\mu}^\alpha_\beta(f) \;=\; \left.\frac{\partial\varphi^\alpha(f', f)(x)}{\partial f'^\beta(x')}\right|_{f'=x'} \;=\; \delta^2(x' - f(x))\delta^\alpha_\beta. \tag{3.6}$$

On the other hand the functions $\mu$ in (2.16) and its inverse $\lambda$ have local forms,

$$\mu^\alpha_\beta(f) = \left.\frac{\partial\varphi^\alpha(f, f')}{\partial f'^\beta(x')}\right|_{f'=x'} \;=\; \partial_\beta f^\alpha(x)\delta(x' - x), \tag{3.7}$$

$$\lambda^\beta_\alpha(f) \;=\; A^{-1\beta}_\alpha(x)\delta(x' - x) \;=\; \frac{1}{\Delta^f}\epsilon^{\beta\gamma}(\partial_\gamma f^\delta)\epsilon_{\delta\alpha}\delta(x' - x). \tag{3.8}$$

In order to find the expression for the WZ term we consider an interpolating variable $f^\alpha_t(x)$ satisfying the boundary conditions

$$f^\alpha_0(x) \;=\; x^\alpha \quad (t = 0), \qquad f^\alpha_1(x) \;=\; f^\alpha(x) \quad (t = 1). \tag{3.9}$$

The finite transformation of the curvature in the anomaly (3.1), which is evaluated by a Weyl invariant combination of zwei-bein $\frac{e_\mu{}^a}{\sqrt{e}}$, is

$$R(\frac{e^{f_t}{}_\mu{}^a(x)}{\sqrt{e^{f_t}(x)}}) \;=\; \Delta^{f_t}(x)[\{\; R(\frac{e_\mu{}^a}{\sqrt{e}}) \;+\; \Box \ln \Delta^{f_t}(F_t(x))\}\Big|_{x=f_t(x)}], \tag{3.10}$$

where $F^\alpha_t(\hat{x})$ is an inverse function of $f^\alpha_t(x) = \hat{x}^\alpha,\ i.e.\ x^\alpha = F^\alpha_t(f_t(x))$. $\Box$ is defined as $\Box \equiv \partial_\alpha(eg^{\alpha\beta}\partial_\beta)$.

Now we are ready to calculate the WZ term (2.24)

$$\begin{aligned}\mathcal{M}_1(\phi, f) =& -i\int_0^1 dt\, \mathcal{A}_\alpha(F(\phi, f_t))\lambda^\alpha{}_\beta(f_t, \phi)(\partial_t f^\beta_t) \\ = -i\int_0^1 dt \int d^2x\ \Delta^{f_t}[\ R(\frac{e_\mu{}^a}{\sqrt{e}}) \;+\; \Box \ln \Delta^{f_t}(F_t(x))]_{x=f_t(x)}\ \partial_\alpha\ [A^{-1\alpha}_{\beta t}(x)\ \dot{f}^\beta_t(x)],\end{aligned} \tag{3.11}$$

where $\lambda^\beta_\gamma$ is given in (3.8). We change the integration variable from $x^\alpha$ to $\hat{x}^\alpha = f^\alpha_t(x)$ ,it becomes

$$\begin{aligned}\mathcal{M}_1(\phi, f) = -i\int_0^1 dt \int d^2\hat{x}\ [\ R(\frac{e_\mu{}^a}{\sqrt{e}}) \;+\; \Box \ln \Delta^{f_t}(F_t)]_{x=\hat{x}} \\ \times [\partial_\alpha\ \{A_{\beta t}{}^{-1\alpha}(x)\ \dot{f}^\beta_t(x)\}]_{x=F_t(\hat{x})}.\end{aligned} \tag{3.12}$$

The last factor in the integrand is expressed as a total derivative with respect to $t$

$$[\partial_\beta\{A^{-1\beta}_{\alpha t}\dot{f}^\alpha_t\}]_{x=F_t(\hat{x})} \;=\; \partial_t[\ln(\Delta^{f_t})_{x=F_t(\hat{x})}]. \tag{3.13}$$

Defining the expression appearing in the r.h.s. as $\Theta_t(\hat{x})$;

$$\Theta_t(\hat{x}) = \ln \Delta^{f_t}(x)\Big|_{x=F_t(\hat{x})} \tag{3.14}$$

it becomes

$$\mathcal{M}_1(\phi, f) \;=\; -i\int_0^1 dt \int d^2\hat{x}\ [\ R(\frac{e_\mu{}^a(\hat{x})}{\sqrt{e(\hat{x})}}) \;+\; \Box_{\hat{x}}\Theta_t(\hat{x})]\partial_t\Theta_t(\hat{x}). \tag{3.15}$$

The $t$ integration in the WZ term is performed and we obtain

$$\mathcal{M}_1(\phi, f) \;=\; -\,i\int d^2x\ [\ R(\frac{e_\mu{}^a}{\sqrt{e}})\ \Theta \;-\; \frac{1}{2}\{e\ g^{\alpha\beta}\ \partial_\beta\ \Theta\ \partial_\alpha\ \Theta\}], \tag{3.16}$$

where we have used the boundary condition (3.9); $\Theta_{t=0}(\hat{x}) = 0$ and we call $\Theta_{t=1}(\hat{x})$ as $\Theta(\hat{x})$;

$$\Theta(\hat{x}) \;\equiv\; \ln \Delta^f(x)\Big|_{x=F(\hat{x})}. \tag{3.17}$$

The WZ term (3.16) is equivalent to one found in [30].

We will make several comments. The variable $\Theta(x)$ is a non-local function of $f^\alpha(x)$. The WZ term, which is non-local as a function of $f^\alpha(x)$, can be expressed only in term of one variable $\Theta(x)$ and becomes a local function of $\Theta(x)$.

The transformation properties of $f^\alpha(x)$ is determined by (2.33). Now

$$\tilde{\mu}'^{\alpha}_{I} \;=\; 0, \quad \xi^\alpha_A \;=\; \mu^\alpha_\nu Z^\nu_A \;=\; \partial_\beta f^\alpha(x)\epsilon^{\beta\gamma}\partial_\gamma. \tag{3.18}$$

Since $\tilde{\mu}^\alpha_\beta$ given in (3.6) is non-local, $f^\alpha(x)$ transforms non-locally as

$$\delta f^\alpha(x) \;=\; -\,\tilde{\mu}'^{\alpha}_{\mu}(f)\;\epsilon^\mu \;+\; \xi^\alpha_A(f)\sigma^A \;=\; -\,\epsilon^\alpha(f(x)) + \partial_\beta f^\alpha(x)\epsilon^{\beta\gamma}\partial_\gamma\sigma(x), \tag{3.19}$$

where $\sigma$ is a parameter of the additional gauge symmetry of the WZ action. The transformation of $\Theta$ is given using that of $f^\alpha$ (3.19) and has a local form

$$\delta\Theta(x) \;=\; \partial_\alpha\Theta(x)\epsilon^\alpha(x) - \partial_\alpha\epsilon^\alpha(x). \tag{3.20}$$

That is it transforms as a scalar under SDiff satisfying $\partial_\alpha\epsilon^\alpha = 0$. The WZ term (3.16) in terms of $\Theta$ is manifestly gauge invariant under $\sigma-$ transformations due to the invariance of the $\Theta$.

Note that the inhomogeneous transformation of $\Theta$ suggests, according to the non-linear realization procedure, that $\Theta$ is an element of the coset $\frac{Diff_2}{SDiff}$. In fact, suppose $f_1^\alpha(x)$ and $f_2^\alpha(x)$ belong to the equivalent class. That is they are connected by a SDiff transformation by $f^\alpha(x)$;

$$f_1^\alpha(x) \;=\; f_2^\alpha(f(x)) \qquad \text{with} \qquad \Delta^f(x) \;=\; 1. \tag{3.21}$$

Their determinants are related by $\Delta^{f_1}(x) = \Delta^{f_2}(f(x))$. The definition of $\Theta$, (3.17), enables us to show that $\Theta$ is a representative of the coset

$$\begin{aligned}\Theta_2(\hat{x}) \;&=\; \ln\Delta^{f_2}(x)\Big|_{x=F_2(\hat{x})} \;=\; \ln\Delta^{f_1}(x)\Big|_{f(x)=F_2(\hat{x})} \\ &=\; \ln\Delta^{f_1}(x)\Big|_{x=F(F_2(\hat{x}))} \;=\; \ln\Delta^{f_1}(x)\Big|_{x=F_1(\hat{x})} \;=\; \Theta_1(\hat{x}).\end{aligned} \tag{3.22}$$

That is $\Theta_1(\hat{x})$ and $\Theta_2(\hat{x})$ have the same value at identical point $\hat{x}$.

Note also that by construction the WZ term satisfies the cocycle condition. It is manifest if it is expressed in terms of finite function $f^\alpha$. However it is not apparent once it is expressed in terms of $\Theta$ as (3.16). The one-cocycle condition is now expressed as

$$\mathcal{M}_1(\phi, \varphi(\Theta, \Theta')) \;=\; \mathcal{M}_1(F(\phi,\Theta),\Theta') \;+\; \mathcal{M}_1(\phi,\Theta). \tag{3.23}$$

Since we have the relation between $\Theta$ and $f^\alpha(x)$ in (3.17), we can find the composition function of $\Theta$'s using the composition rule (3.5), $f''(x) = f(f'(x))$,

$$\Theta''(\hat{x}) \;\equiv\; \varphi(\Theta,\Theta') \;=\; \Theta(\hat{x}) \;+\; [\Theta'(x')]_{x'=F(\hat{x})}. \tag{3.24}$$

The composition law has non-local form since $\Theta'$ is evaluated at $x' = F(\hat{x}) \neq \hat{x}$, from which we can check the cocycle condition (3.23) explicitly.

The extended action $\tilde{S}$ in (2.26) of this system is

$$\tilde{S} \;=\; S \;+\; \int dx[\; f^*_\alpha\{-C^\alpha(f(x)) + \partial_\beta f^\alpha\epsilon^{\beta\gamma}\partial_\gamma b\;\} \;+\; \frac{1}{2}b^*\epsilon^{\alpha\beta}\partial_\alpha b\partial_\beta b\;], \tag{3.25}$$

where the first term $S$ is the solution of the CME (2.25) given by

$$\begin{aligned} S \quad &= S_0 \ + \ \int dx\{X^* \ \partial_\alpha X \ C^\alpha \ + \ e^{\alpha}_{\ a}{}^{*} \ (\partial_\beta e_\alpha{}^a \ C^\beta + \epsilon^a{}_b \ e_\alpha{}^b \ C_L + e_\alpha{}^a \ C_W) \\ &+ C_\alpha{}^* \ \partial_\beta C^\alpha \ C^\beta \ + \ C_L{}^* \ \partial_\alpha C_L \ C^\alpha + \ C_W{}^* \ \partial_\alpha C_W \ C^\alpha \ \}, \end{aligned} \tag{3.26}$$

and $b$ is the ghost for the additional symmetry of the WZ action; SDiff. It is non-local in terms of $f^\alpha(x)$. Since the non-anomalous transformations are closed in this case the extended action $\tilde{S}$ satisfies the CME; $(\tilde{S}, \tilde{S}) \ = \ 0$.

The anomalous degree of freedom of $f^\alpha$ has been expressed as $\Theta$. We can show that the other (non-anomalous) degree of freedom, say $\Theta'$, and the additional ghost $b(x)$ are transformed into a form of trivial non-minimal pairs. It is performed by making a canonical transformation. $\Theta'$ can be any function $w(f)$ as long as it is independent of $\Theta(f)$. The generating function is

$$\begin{aligned} W \quad &= \int dx \ [ \ \Theta^*(x) \ln(\Delta^f)_{x=F(x)} \ + \ \Theta'^*(x) w(f(x)) \\ &+ \tilde{b}^*(x) \ \frac{\partial w(f)}{\partial f^\alpha} \ \{-C^\alpha(f(x)) \ + \ \partial_\beta f^\alpha \epsilon^{\beta\gamma} \partial_\gamma b(x) \ \} \ + \ \tilde{C}^*_\alpha C^\alpha \ ], \end{aligned} \tag{3.27}$$

It defines new fields,

$$\begin{aligned} \Theta(x) \ &= \ \ln(\Delta^f)_{x=F(x)}, \qquad \Theta'(x) \ = \ w(f(x)), \\ \tilde{b}(x) \ &= \ w_\alpha(f) \ \{-C^\alpha(f(x)) \ + \ \partial_\beta f^\alpha \epsilon^{\beta\gamma} \partial_\gamma b(x) \ \}, \quad \tilde{C}^\alpha(x) \ = \ C^\alpha(x) \end{aligned} \tag{3.28}$$

and the corresponding anti-fields. In terms of new variables, all $\tilde{b}^*$ terms cancel out and we obtain

$$\tilde{S} \ = \ \hat{S} \ + \ \int dx \ [ \ \Theta^* \{ \partial_\alpha \Theta \tilde{C}^\alpha - \partial_\alpha \tilde{C}^\alpha \ \} \ + \ \Theta'^* \tilde{b} \ ], \tag{3.29}$$

where $\hat{S}$ is $S$ in which all fields and anti-fields are replaced by corresponding new ones. The second term tells the transformation of $\Theta$ (3.20). The third term shows that $(\Theta', \Theta'^*)$ and $(\tilde{b}, \tilde{b}^*)$ are non-minimal pairs and are cohomologically trivial. Note the extended action became local one as the result of the non-local canonical transformation.

# 4 Summary

In this paper we have discussed in a unified way the WZ terms for: gauged effective field theories associated with a spontaneously broken global symmetry, and anomalous gauge theories. The formalism is applicable for the quasigroup case in which each the anomalous and non-anomalous transformations do not form closed subgroups. In general we should introduce the NG fields associated with the non-anomalous transformations . In this case the WZ terms have (hidden) gauge symmetries in such a way to kill the extra NG fields. Our procedure can be seen as a generalization of the non-linear realization method of Lie groups.

As an example, we have considered two dimensional gravity with anomalous diffeomorphism , and obtained the local WZ term described in terms of the coset coordinate of $\frac{Diff_2}{SDiff}$.

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# A Quasigroup Structures

Here we give a brief summary of quasigroup structures and some definitions [20]. Let us consider the action of quasigroup $\mathcal{G}$ which is locally described by a set of parameters $\theta^\mu$ on a manifold $\mathcal{M}$ parametrized by the classical fields $\phi^i$

$$\begin{aligned} F : \mathcal{M} \times \mathcal{G} &\to \mathcal{M} \\ (\phi^i,\ \theta^\mu) &\mapsto F^i(\phi,\theta). \end{aligned} \tag{A.1}$$

We assume it is closed and irreducible for simplicity. The relevant properties are invariance of the action;

$$\mathcal{S}_0(F(\phi,\theta)) = \mathcal{S}_0(\phi) \tag{A.2}$$

and the composition law;

$$F^i(F(\phi,\theta),\theta') = F^i(\phi,\varphi(\theta,\theta';\phi)), \tag{A.3}$$

where $\varphi^\mu(\theta,\theta';\phi)$ represents the composition function of the parameters of quasigroup $\mathcal{G}$.

The structure functions appearing in the solution of the classical master equation (2.25) are directly related to these functions,

$$R^i{}_\mu(\phi) \ := \ \left.\frac{\partial F^i(\phi,\theta)}{\partial\theta^\mu}\right|_{\theta=0}, \qquad T^\rho_{\mu\nu}(\phi) := -\left(\frac{\partial^2\varphi^\rho(\theta,\theta';\phi)}{\partial\theta^\mu\partial\theta'^\nu} - (\nu\leftrightarrow\mu)\right)_{\theta=\theta'=0}. \tag{A.4}$$

They satisfy

$$R^j{}_{[\mu}(\phi) R^i{}_{\nu],j}(\phi) + T^\rho_{\mu\nu}(\phi) R^i{}_\rho(\phi) \ = \ 0, \tag{A.5}$$

$$\sum_{P\in\mathrm{Perm}[\mu\nu\rho]} (-1)^P \big(\ T^\delta_{\nu\rho,j}(\phi)\ R^j{}_\mu(\phi)\ + T^\delta_{\sigma\rho}(\phi)\ T^\sigma_{\mu\nu}(\phi)\ \big) \ = \ 0. \tag{A.6}$$

The latter is the generalized Jacobi identity.

From the composition law (A.3) we can obtain an analogue of the Lie equation by multiplying an operator $\left.\frac{\partial}{\partial\theta'}\right|_{\theta'=0}$ on it,

$$\frac{\partial F^i(\phi,\theta)}{\partial\theta^\sigma} = R^i_\nu(F(\phi,\theta))\lambda^\nu{}_\sigma(\theta,\phi), \tag{A.7}$$

where $\lambda^{\nu}{}_{\sigma}(\theta,\phi)$ is the inverse matrix of

$$\mu^{\sigma}{}_{\nu}(\theta,\phi) = \left.\frac{\partial\varphi^{\sigma}(\theta,\theta',\phi)}{\partial\theta'^{\nu}}\right|_{\theta'=0}. \tag{A.8}$$

On the other hand if we operate $\frac{\partial}{\partial\theta}\Big|_{\theta=0}$ on (A.3) we have

$$\frac{\partial F^i(\phi,\theta)}{\partial\phi^k}\; R^k_{\sigma}(\phi) \;=\; \frac{\partial F^i(\phi,\theta)}{\partial\theta^{\nu}}\; \tilde{\mu}^{\nu}{}_{\sigma}(\theta;\phi), \tag{A.9}$$

where

$$\tilde{\mu}^{\nu}{}_{\sigma}(\theta,\phi) := \left.\frac{\partial\varphi^{\nu}(\theta',\theta;\phi)}{\partial\theta'^{\sigma}}\right|_{\theta'=0}. \tag{A.10}$$

Some useful formulas are

$$\begin{aligned}
\tilde{\mu}^{\rho}_{\mu}\; D_{\rho}\; \tilde{\mu}^{\sigma}_{\nu} - \tilde{\mu}^{\rho}_{\nu}\; D_{\rho}\; \tilde{\mu}^{\sigma}_{\mu} \;&=\; \tilde{\mu}^{\sigma}_{\rho}\; T^{\rho}_{\mu\nu}(\phi), && \text{(A.11)}\\
\mu^{\rho}_{\mu}\; \partial_{\rho}\; \mu^{\sigma}_{\nu} - \mu^{\rho}_{\nu}\; \partial_{\rho}\; \mu^{\sigma}_{\mu} \;&=\; -\mu^{\sigma}_{\rho}\; T^{\rho}_{\mu\nu}(F(\phi,\theta)), && \text{(A.12)}\\
\tilde{\mu}^{\rho}_{\mu}\; D_{\rho}\; \mu^{\sigma}_{\nu} - \mu^{\rho}_{\nu}\; \partial_{\rho}\; \tilde{\mu}^{\sigma}_{\mu} \;&=\; 0, && \text{(A.13)}
\end{aligned}$$

where $\tilde{\lambda}^{\sigma}_{\rho}$ is the inverse of $\tilde{\mu}^{\rho}_{\sigma}$ and $D_{\rho} \;\equiv\; \frac{\partial}{\partial\theta^{\rho}} - R^j_{\sigma}(\phi)\; \tilde{\lambda}^{\sigma}_{\rho}\; \frac{\partial}{\partial\phi^j} \;\equiv\; \partial_{\rho} - R^j_{\sigma}(\phi)\; \tilde{\lambda}^{\sigma}_{\rho}\; \partial_j$.

# B The Extended Action

In order to study the gauge structure of the extended formalism of fields $\phi$ and $\theta$ in more detail it is useful to introduce a field antifield formalism and the classical master equation (CME) that encodes the gauge structure of the theory [33][34]. The solution of CME in the original space of $(\phi^j, c^{\mu})$ and their anti-fields $(\phi^*_j, c^*_{\mu})$ is

$$S \;=\; S_0(\phi^j) \;+\; \phi^*_j\; R^j_{\mu}(\phi)\; c^{\mu} \;+\; \frac{1}{2}c^*_{\mu}\; T^{\mu}_{\rho\sigma}(\phi)\; c^{\sigma} c^{\rho} \tag{B.1}$$

and satisfies, $(S,S)=0$. Here we use (A.5) and (A.6).

We first construct an action in the extended phase space, by introducing ghosts $b^{\mu}$,

$$S_1 \;=\; S \;+\; \theta^*_{\mu}\; \{-\tilde{\mu}^{\mu}_{\nu}(\theta;\phi)\; c^{\nu} \;+\; \mu^{\mu}_{\nu}(\theta;\phi) b^{\nu}\} \;+\; \frac{1}{2} b^*_{\rho} T^{\rho}_{\mu\nu}(F(\phi,\theta)) b^{\nu} b^{\mu}, \tag{B.2}$$

which satisfies CME $(S_1,S_1)=0$ using (A.11)(A.12)(A.13).

However the WZ term is not invariant under all $b^{\mu}$ transformations but

$$\delta_{\sigma}\mathcal{M}_1 \;= -\delta_{\sigma}\mathcal{M}^{non}(F) = -\mathcal{A}_{\rho}(F)\lambda^{\rho}_{\mu}\mu^{\mu}_{\nu} b^{\nu} = -\tilde{\mathcal{A}}_a(F)\tilde{Z}^a_{\nu}(F) b^{\nu}. \tag{B.3}$$

Thus the WZ term is invariant if $b^{\nu}$ take the form

$$b^{\nu} \;=\; Z^{\nu}_A(F)\; \tilde{b}^A. \tag{B.4}$$

Therefore we make a canonical transformation by

$$W \;=\; b^*_\mu\; Z^\mu_{\tilde\nu}(F)\; \tilde b^{\tilde\nu} \;+\; \phi^*_j\; \tilde\phi^j \;+\; \theta^*_\mu\; \tilde\theta^\mu \;+\; c^*_\mu\; \tilde c^\mu, \tag{B.5}$$

where tilded indices, $\tilde\nu$, runs through non anomalous ones, $A$, and anomalous ones, $a$. It gives

$$b^\mu \;=\; Z^\mu_{\tilde\nu}(F)\; \tilde b^{\tilde\nu} \;\equiv\; Z^\mu_A(F)\; \tilde b^A \;+\; Z^\mu_a(F)\; \tilde b^a, \qquad \tilde b^*_{\tilde\nu} \;=\; b^*_\mu\; Z^\mu_{\tilde\nu}(F) \tag{B.6}$$

$$\tilde\phi^*_j \;=\; \phi^*_j \;+\; b^*_\mu \tilde b^{\tilde\mu} \frac{\partial Z^\mu_{\tilde\mu}(F)}{\partial F^i}\frac{\partial F^i}{\partial \phi^j}, \qquad \tilde\theta^*_\rho \;=\; \theta^*_\rho \;+\; b^*_\mu \tilde b^{\tilde\mu} \frac{\partial Z^\mu_{\tilde\mu}(F)}{\partial F^i}\frac{\partial F^i}{\partial \theta^\rho}. \tag{B.7}$$

The action $S_1$ becomes

$$S_1 \;=\; \hat S \;+\; \tilde\theta^*_\mu\; \{ -\tilde\mu^\mu_\nu(\theta;\phi)\; c^\nu \;+\; \mu^\mu_\nu(\theta;\phi) Z^\nu_{\tilde\nu}(F)\tilde b^{\tilde\nu}\} \;+\; \frac{1}{2}\tilde b^*_{\tilde\rho} \tilde T^{\tilde\rho}_{\tilde\mu\tilde\nu}(F(\phi,\theta))\tilde b^{\tilde\nu}\tilde b^{\tilde\mu},, \tag{B.8}$$

where $\hat S$ is $S$ in which the old variables are replaced by the corresponding the new ones and

$$\tilde T^{\tilde\rho}_{\tilde\mu\tilde\nu}(\phi) \;=\; \tilde Z^{\tilde\rho}_\rho\; T^\rho_{\mu\nu}\; Z^\mu_{\tilde\mu}\; Z^\nu_{\tilde\nu} \;-\; \tilde Z^{\tilde\rho}_\rho\; \frac{\partial Z^\rho_{[\tilde\nu}}{\partial \phi^i}\; R^i_\sigma\; Z^\sigma_{\tilde\mu]}. \tag{B.9}$$

It satisfies the generalized Jacobi identity corresponding to (A.6).

$$\sum_{P\in \mathrm{Perm}[\mu\nu\rho]} (-1)^P (\; \tilde T^\delta_{\nu\rho,j}\; R^j{}_\sigma\; Z^\sigma_\mu \;+ \tilde T^\delta_{\sigma\rho}\; \tilde T^\sigma_{\mu\nu}\;) \;=\; 0. \tag{B.10}$$

Since we made a canonical transformation the $S_1$ satisfies the CME also.

We are interested in an extended formalism which incorporates the gauge invariances of the WZ term we must impose $\tilde b^a = 0$. It is consistent only on the WZ equation since

$$\delta \tilde b^a \;\Big|_{\tilde b^a = 0} \;=\; \frac{1}{2}\tilde T^a_{AB}(F)\tilde b^B \tilde b^A \tag{B.11}$$

and $\tilde T^a_{AB}(F)$ vanishes on shell of WZ equation of motion, see (2.31) and (2.32). Correspondingly the action

$$\tilde S \;=\; S_1|_{\tilde b^a=0} \;=\; \hat S \;+\; \tilde\theta^*_\mu\; \{ -\tilde\mu^\mu_\nu(\theta;\phi)\; c^\nu \;+\; \mu^\mu_\nu(\theta;\phi) Z^\nu_A(F)\tilde b^A\} \;+\; \frac{1}{2}\tilde b^*_A \tilde T^A_{BD}(F(\phi,\theta))\tilde b^D\tilde b^B, \tag{B.12}$$

satisfies CME on shell of the WZ equation.

$$(\tilde S, \tilde S) \;=\; -\{\tilde\theta^*_\mu\; \mu^\mu_\nu\; Z^\nu_a(F) \;+\; \tilde b^*_A\; \tilde T^A_{aB}(F)\; \tilde b^B\}\; \tilde T^a_{DE}(F)\tilde b^E \tilde b^D, \tag{B.13}$$

where we have used (B.10).

Rewriting new variables by omitting *tildes* ,e.g. $\tilde b^A \;\to\; b^A$, we find the extended action (2.26) and the on shell nilpotency (2.28) in the section 2.2.

# C Reduction of variables in $\mathcal{G} - \mathcal{G}_0$

When there is a subgroup $\mathcal{G}_0$ whose algebra includes all anomalous generators we can reduce the extra variables $\theta^I$'s of $\mathcal{G} - \mathcal{G}_0$. It is corresponding to define the Dirac brackets in the canonical constrained systems.

First we introduce the extra variable $\theta^\mu$ for all transformations in $\mathcal{G}$. The extended action $\tilde{S}$ of the system is (2.26);

$$\tilde{S} \;=\; S \;+\; \theta^*_\mu \,\{-\tilde{\mu}^\mu_\nu(\theta,\phi)\; c^\nu \;+\; \xi^\mu_{\tilde{A}}(\theta,\phi) b^{\tilde{A}}\} \;+\; \frac{1}{2} b^*_{\tilde{A}} \tilde{T}^{\tilde{A}}_{\tilde{B}\tilde{D}}(F) b^{\tilde{D}} b^{\tilde{B}}, \tag{C.1}$$

where indices $\tilde{A}, \tilde{B}, ...$ are those of non-anomalous transformations ; $A, B, ...$ of $\mathcal{G}$ and $I, J, K, ...$ of $\mathcal{G} - \mathcal{G}_0$.

To reduce the trivial non-anomalous variables $\theta^I$ of $\mathcal{G} - \mathcal{G}_0$ we impose

$$\theta^I \;=\; 0 \tag{C.2}$$

and require

$$0 \;=\; \delta\theta^I\Big|_{\theta^I\;=\;0} \;=\; \{-\tilde{\mu}^I_\nu\; c^\nu \;+\; \xi^I_A\; b^A + \; \xi^I_J\; b^J\}\Big|_{\theta^I\;=\;0}. \tag{C.3}$$

Since $\mathcal{G}_0$ is the subgroup it holds, for indices $\alpha$ (A and a) of $\mathcal{G}_0$ and for $\theta^I \;=\; 0$

$$\mu^I_\alpha \;=\; \tilde{\mu}^I_\alpha \;=\; \lambda^I_\alpha \;=\; \tilde{\lambda}^I_\alpha \;=\; 0. \tag{C.4}$$

Also $\theta^I$ transformations of $\mathcal{G} - \mathcal{G}_0$ are non-anomalous transformations decoupling from $\theta^\alpha$ of $\mathcal{G}_0$ we can take $Z^\nu_I = \delta^\nu_I$ and $Z^J_\alpha = 0$ thus $\xi^I_A = \mu^I_\nu Z^\nu_A = 0$ and $\xi^I_J = \mu^I_\nu Z^\nu_J = \mu^I_J$. The condition (C.3) determines $b^I$;

$$\tilde{b}^J \;\equiv\; b^J - M^J_K c^K \;=\; 0, \qquad M^J_K \;\equiv\; \lambda^J_L\; \tilde{\mu}^L_K\Big|_{\theta^I\;=\;0}. \tag{C.5}$$

We define a reduced subspace (denoted with $\star$) by $\theta^I = \tilde{b}^I = 0$. It is closed since $\delta^2\theta^I \;=\; 0$ due to (C.4).

In order to reduce these variables explicitly we make a canonical transformation generated by

$$W \;=\; \tilde{b}^*_J(b^J - M^J_L c^L) \;+\; \tilde{\phi}^*_j \phi^j \;+\; \tilde{c}^*_\mu c^\mu \;+\; \tilde{\theta}^*_\mu \theta^\mu \;+\; \tilde{b}^*_\alpha b^\alpha. \tag{C.6}$$

It defines new field

$$\tilde{b}^J \;=\; b^J - M^J_L c^L \tag{C.7}$$

and anti-fields by

$$c^*_L \;=\; \tilde{c}^*_L - \tilde{b}^*_J M^J_L, \qquad \phi^*_j \;=\; \tilde{\phi}^*_j - \tilde{b}^*_J M^J_{L,j} c^L, \qquad \theta^*_\alpha \;=\; \tilde{\theta}^*_\alpha - \tilde{b}^*_J M^J_{L,\alpha} c^L. \tag{C.8}$$

Other variables are unchanged.

The BRST transformation of functions in reduced space is generated by $S^\star$,

$$\delta f(\tilde{\Phi},\tilde{\Phi}^*)\Big|_\star \;=\; (f(\tilde{\Phi},\tilde{\Phi}^*),\tilde{S})\Big|_\star \;=\; (f(\tilde{\Phi},\tilde{\Phi}^*),S^\star), \qquad S^\star \;\equiv\; \tilde{S}\Big|_\star. \tag{C.9}$$

It is also shown that

$$(\tilde{S},\tilde{S})\Big|_{\star} \;=\; (S^{\star},S^{\star}). \tag{C.10}$$

Thus $S^{\star}$ plays the role of $\tilde{S}$ in the reduced space. It is

$$S^{\star} \;=\; S_0(\tilde{\phi}) \;+\; \tilde{\phi}^{*}_{j}\, R^{j}_{\mu}(\tilde{\phi})\, \tilde{c}^{\mu} \;+\; \frac{1}{2}\tilde{c}^{*}_{\mu}\, T^{\mu}_{\rho\sigma}(\tilde{\phi})\, \tilde{c}^{\sigma}\tilde{c}^{\rho} \;+\; \tilde{\theta}^{*}_{\alpha}\, \{-\tilde{\mu}'^{\alpha}_{\nu}\, \tilde{c}^{\nu} \;+\; \xi^{\alpha}_{A}\, \tilde{b}^{A}\} \;+$$
$$\frac{1}{2}\tilde{b}^{*}_{D}\{T^{D}_{AB}(F)\tilde{b}^{B}\tilde{b}^{A} \;+\; 2\, T^{D}_{IB}(F)\tilde{b}^{B}M^{I}_{L}\tilde{c}^{L} \;+\; T^{D}_{IJ}(F)M^{J}_{K}\tilde{c}^{K}M^{I}_{L}\tilde{c}^{L}\}. \tag{C.11}$$

Note it does not depend on $\tilde{b}^{*}_{J}$ and the transformation of $\theta^{\alpha}$ has been changed as

$$\delta\tilde{\theta}^{\alpha} \;=\; \{-\tilde{\mu}'^{\alpha}_{\nu}\, \tilde{c}^{\nu} \;+\; \xi^{\alpha}_{A}\, \tilde{b}^{A}\}, \qquad \tilde{\mu}'^{\alpha}_{I} \;\equiv\; \tilde{\mu}^{\alpha}_{I} - \mu^{\alpha}_{J}M^{J}_{I}, \quad \tilde{\mu}^{\alpha'}_{\beta} \;\equiv\; \tilde{\mu}^{\alpha}_{\beta}. \tag{C.12}$$

$S^{\star}$ also satisfies CME only on shell of the WZ equation of motion

$$(S^{\star},\; S^{\star}) \;=\; (\tilde{S},\tilde{S})\Big|_{\star} \;=\; -\,\{\theta^{*}_{\alpha}\mu^{\alpha}_{\nu}Z^{\nu}_{a}(F) \;+\; \tilde{b}^{*}_{A}\; T^{A}_{a\tilde{B}}(F)\; b^{\tilde{B}}\}\; T^{a}_{\tilde{D}\tilde{E}}(F)b^{\tilde{E}}b^{\tilde{D}}, \tag{C.13}$$

where $b^{\alpha} \;=\; \tilde{b}^{\alpha}$ and $b^{I} \;=\; M^{I}_{J}\tilde{c}^{J}$.

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